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A note on definition of matrix convex functions ☆

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Abstract

We prove that a real-valued function f defined on an interval S in $\mathbf{R}$ is matrix convex if and only if for any natural k, for all families of positive operators $\{A_i\}_{i=1}^k$ in a finite-dimensional Hilbert space, such that $\sum_{i=1}^k A_i = 1$, and arbitrary numbers $x_i \in S$, the inequality

$$f\left(\sum_{i=1}^k x_i A_i\right) \leqslant \sum_{i=1}^k f(x_i) A_i$$

holds true.

Keywords: Matrix convex function; The Neumark theorem

The aim of this note is to add one more characterization to the list of known characterizations of matrix and operator convex functions (see, e. g., [1,3,4]).

In what follows, S stands for an interval in $\mathbf{R}$ of an arbitrary type. For a function $f: S \to \mathbf{R}$ and a self-adjoint operator X in a finite-dimensional Hilbert space H with spectrum in S, the value $f(X)$ is defined by the spectral theorem. The identity operator in H is denoted by 1_H.

Theorem. *For a function $f: S \to \mathbf{R}$, the following conditions are equivalent*:

(i) *f is matrix convex*, *i.e.*,

$$f(\alpha X + (1-\alpha)Y) \leqslant \alpha f(X) + (1-\alpha) f(Y)$$

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